Chapter 8

Expanding the Equation Horizon

In This Chapter

- Going to a higher power with cubic equations
- Using synthetic division to solve polynomial equations
- Rounding up radicals and solving their problems

In Chapters 6 and 7, you find out how to solve equations with powers of 1 and 2, and this seems to be enough to get through most of the applications. But every once in a while, you'll be thrown a curve with an equation of a degree higher than 2 or an equation with a radical in it or a fractional degree in it. No need to panic. You can deal with these rogue equations in many ways, and in this chapter, I tell you what the most efficient ways are. One common thread you'll see in solving these equations is a goal to set the expression equal to 0 so you can use the multiplication property of zero (see Chapter 7) to find the solution.

Queuing Up to Cubic Equations

Cubic equations contain a variable term with a power of 3 but no power higher than 3. In these equations, you can expect to find up to three different solutions, but there may not be as many as three. Can you assume that fourth-degree equations could have as many as four solutions and fifth-degree equations . . . ? Yes, indeed you can — this is the general rule. The degree can tell you what the *maximum* number of solutions is. Although the number of solutions *may* be less than the number of the degree, there won't be any more solutions than that number.

Solving perfectly cubed equations

If a cubic equation has just two terms and they're both perfect cubes, then your task is easy. The sum or difference of perfect cubes can be factored into two factors with only one solution. The first factor, or the *binomial,* gives you a solution. The second factor, the *trinomial,* does not give you a solution.

If $x^3 - a^3 = 0$, then $x^3 - a^3 = (x - a)(x^2 + ax + a^2) = 0$ and $x = a$ is the only solution. Likewise, if $x^3 + a^3 = 0$, then $(x + a)(x^2 - ax + a^2) = 0$ and $x = -a$ is the only solution. The reason you have only one solution for each of these cubics is because $x^2 + ax + a^2 = 0$ and $x^2 - ax + a^2 = 0$ have no real solutions. The trinomials can't be factored, because the quadratic formula gives you imaginary solutions.

Solve for y in $27y^3 + 64 = 0$ using factoring.

The factorization here is $27y^3 + 64 = (3y + 4)(9y^2 - 12y + 16)$. The first factor offers a solution, so set $3y + 4$ equal to 0 to get $3y = -4$ or $y = -\frac{4}{3}$.

Solve for a in $8a^3 - (a - 2)^3 = 0$ using factoring.

The factorization here works the same as factorizations of the difference between perfect cubes. It's just more complicated because the second term is a binomial:

$$8a^3 - (a - 2)^3 = [2a - (a - 2)][4a^2 + 2a(a - 2) + (a - 2)^2] = 0$$

Simplify inside the first bracket by distributing the negative and you get

$$[2a - (a - 2)] = [2a - a + 2] = [a + 2]$$

Setting the first factor equal to 0, you get

$$\begin{aligned} a + 2 &= 0 \\ a &= -2 \end{aligned}$$

As usual, the second factor doesn't give you a real solution, even if you distribute, square the binomial, and combine all the like terms.

Going for the greatest common factor

Another type of cubic equation that's easy to solve is one in which you can factor out a variable greatest common factor (GCF), leaving a second factor that is linear or quadratic (first or second degree). You apply the multiplication property of zero (MPZ) and work to find the solutions — usually three of them.

Factoring out a first-degree variable greatest common factor

When the terms of a three-term cubic equation all have the same first-degree variable as a factor, then factor that out. The resulting equation will have the variable as one factor and a quadratic expression as the second factor. The first-degree variable will always give you a solution of 0 when you apply the MPZ. If the quadratic has solutions, you can find them using the methods in Chapter 7.

Solve for x in $x^3 - 4x^2 - 5x = 0$.

1. **Determine that each term has a factor of *x* and factor that out.**

   The GCF is x. Factor to get $x(x^2 - 4x - 5) = 0$.

   You're all ready to apply the MPZ when you notice that the second factor, the quadratic, can be factored. Do that first and then use the MPZ on the whole thing.

2. **Factor the quadratic expression, if possible.**

   $$x(x^2 - 4x - 5) = x(x - 5)(x + 1) = 0$$

3. **Apply the MPZ and solve.**

   Setting the individual factors equal to 0, you get $x = 0$, $x - 5 = 0$, or $x + 1 = 0$. This means that $x = 0$ or $x = 5$ or $x = -1$.

4. **Check the solutions in the original equation.**

   If $x = 0$, then $0^3 - 4(0)^2 - 5(0) = 0 - 0 - 0 = 0$.

   If $x = 5$, then $5^3 - 4(5)^2 - 5(5) = 125 - 4(25) - 25 = 125 - 100 - 25 = 0$.

If $x = -1$, then $(-1)^3 - 4(-1)^2 - 5(-1) = -1 - 4(1) + 5 = -1 - 4 + 5 = 0$.

All three work!

Factoring out a second-degree greatest common factor

Just as with first-degree variable GCFs, you can also factor out second-degree variables (or third-degree, fourth-degree, and so on). Factoring leaves you with another expression that may have additional solutions.

Solve for w in $w^3 - 3w^2 = 0$.

1. **Determine that each term has a factor of w^2 and factor that out.**

   Factoring out w^2, you get $w^3 - 3w^2 = w^2(w - 3) = 0$.

2. **Use the MPZ.**

   $w^2 = 0$ or $w - 3 = 0$.

3. **Solve the resulting equations.**

   Solving the first equation involves taking the square root of each side of the equation. This process usually results in two different answers — the positive answer and the negative answer. However, this isn't the case with $w^2 = 0$ because 0 is neither positive nor negative. So there's only one solution from this factor: $w = 0$. (Actually, 0 is a *double* root, because it appears twice.) And the other factor gives you a solution of $w = 3$. So, even though this is a cubic equation, there are only two unique solutions to it.

Grouping cubes

Grouping is a form of factoring that you can use when you have four or more terms that don't have a single GCF. These four or more terms may be grouped, however, when pairs of the terms have factors in common. The method of grouping is covered in Chapter 5. I give you one example here, but turn to Chapter 5 for a more complete explanation.

Solve for x in $x^3 + x^2 - 4x - 4 = 0$.

1. **Use grouping to factor, taking x^2 out of the first two terms and –4 out of the last two terms. Then factor $(x + 1)$ out of the newly created terms.**

   $$x^3 + x^2 - 4x - 4 = x^2(x + 1) - 4(x + 1) = (x + 1)(x^2 - 4) = 0$$

2. **The second factor is the difference between two perfect squares and can also be factored.**

   $$(x + 1)(x^2 - 4) = (x + 1)(x - 2)(x + 2) = 0$$

3. **Solve using the MPZ.**

   $x + 1 = 0$, $x - 2 = 0$, or $x + 2 = 0$, which means that $x = -1$, $x = 2$, or $x = -2$.

There are three different answers in this case, but you sometimes get just one or two answers.

Solving cubics with integers

If you can't solve a third-degree equation by finding the sum or difference of the cubes, factoring, or grouping, you can try one more method that finds all the solutions if they happen to be integers. Cubic equations could have one, two, or three different integers that are solutions. Having all three integral solutions generally only happens if the coefficient (multiplier) on the third-degree term is a 1. If the coefficient on the term with the variable raised to the third power isn't a 1, then at least one of the solutions may be a fraction (not always, but more frequently than not). Synthetic division (see the "Using Synthetic Division" section, later in this chapter) can be used to look for solutions.

Find the solutions for $x^3 - 7x^2 + 7x + 15 = 0$ using the method of integer factors.

To find the solutions when there are all integer solutions, follow these steps:

1. **Write the cubic equation in decreasing powers of the variable. Look for the constant term and list all the numbers that divide that number evenly (its factors).**

Remember to include both positive and negative numbers.

In the cubic equation $x^3 - 7x^2 + 7x + 15 = 0$, the cubic is in decreasing powers, and the constant is 15. The list of numbers that divides 15 evenly is: ± 1, ± 3, ± 5, and ± 15. This is a long list, but you know that somehow or another the factors of the cubic have to multiply to get 15.

2. **Find a number from the list that makes the equation equal 0.**

   Choose a 3 for your first guess. Trying $x = 3$, $(3)^3 - 7(3)^2 + 7(3) + 15 = 27 - 63 + 21 + 15 = 63 - 63 = 0$. It works!

3. **Divide the constant by that number.**

   The answer to that division is your new constant. In the example, divide the original 15 by 3 and get 5. That's your new constant.

4. **Make a list of numbers that divide the new constant evenly.**

   Make a new list for the new constant of 5. The numbers that divide 5 evenly are: ± 1 and ± 5. Four numbers are much nicer than eight.

5. **Find a number from the new list that checks (makes the equation equal 0).**

   Trying $x = 1$, you get $(1)^3 - 7(1)^2 + 7(1) + 15 = 1 - 7 + 7 + 15 = 23 - 7 = 16$. That doesn't work, so try another number from the list.

   Trying $x = 5$, $(5)^3 - 7(5)^2 + 7(5) + 15 = 125 - 175 + 35 + 15 = 175 - 175 = 0$. So, it works.

6. **Divide the new constant by the newest answer.**

   That answer gives you the choices for the last solution.

   Dividing the new constant of 5 by 5, you get 1. The only things that divide that evenly are 1 or -1. Because you already tried the 1, and it didn't work, it must mean that the -1 is the last solution.

   When $x = -1$, you get $(-1)^3 - 7(-1)^2 + 7(-1) + 15 = -1 - 7 - 7 + 15 = 0$.

That does work, of course, so your solutions for $x^3 - 7x^2 + 7x + 15 = 0$ are: $x = 3$, $x = 5$, and $x = -1$. This also means that the factored version of the cubic is $(x - 3)(x - 5)(x + 1) = 0$.

Whew! That's quite a process. But it makes a lot of sense.

Using Synthetic Division

Cubic equations that have nice integer solutions make life easier. But how realistic is that? Many answers to cubic equations that are considered to be rather nice are actually fractions. And what if you want to broaden your horizons beyond third-degree polynomials and try fourth- or fifth-degree equations or higher? Trying out guesses of answers until you find one that works can get pretty old pretty fast.

A method known as *synthetic division* can help out with all these concerns and lessen the drudgery. Synthetic division is a shortcut division process. It takes the coefficients on all the terms in an equation and provides a method for finding the answer to a division problem by only multiplying and adding. It's really pretty neat. I like to use synthetic division to help find both integer solutions and fractional solutions for polynomial equations when it's convenient and shows some promise. Refer to Chapter 5 for more on synthetic division.

Earlier in this chapter, in the "Solving cubics with integers" section, I show you how to choose possible solutions for cubic equations whose lead coefficient is a 1. This section expands your capabilities of finding rational solutions. You see how to solve equations with a degree higher than 3, and you see how to include equations whose lead coefficient is something other than 1.

Here's the general process to use:

1. **Put the terms of the equation in decreasing powers of the variable.**
2. **List all the possible factors of the constant term.**
3. **List all the possible factors of the coefficient of the highest power of the variable (the *lead coefficient*).**

4. **Divide all the factors in Step 2 by the factors in Step 3.**

   This is your list of possible *rational* solutions of the equation.

5. **Use synthetic division to check the possibilities.**

Find the solutions of the equation: $2x^4 + 13x^3 + 4x^2 = 61x + 30$.

1. **Put the terms of the equation in decreasing powers of the variable.**

   $$2x^4 + 13x^3 + 4x^2 - 61x - 30 = 0$$

2. **List all the possible factors of the constant term.**

   The constant term –30 has the following factors: ±1, ±2, ±3, ±5, ±6, ±10, ±15, and ±30.

3. **List all the possible factors of the coefficient of the highest power of the variable (the *lead coefficient*).**

   The lead coefficient 2 has factors ±1 and ±2.

4. **Divide all the factors in Step 2 by the factors in Step 3. This is your list of possible *rational* solutions of the equation.**

   Dividing the factors of –30 by +1 or –1 doesn't change the list of factors. Dividing by +2 or –2 adds fractions when the number being divided is odd — the even numbers just provide values already on the list. So the complete list of possible solutions is: ±1, ±2, ±3, ±5, ±6, ±10, ±15, ±30, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, and $\pm\frac{15}{2}$.

5. **Use synthetic division to check the possibilities.**

   I first try the number 2 as a possible solution. The final number in the synthetic division is the value of the polynomial that you get by substituting in the 2, so you want the number to be 0.

   $$\begin{array}{r|rrrrr} \underline{2} & 2 & 13 & 4 & -61 & -30 \\ & & 4 & 34 & 76 & 30 \\ \hline & 2 & 17 & 38 & 15 & 0 \end{array}$$

   The 2 is a solution, because the final number (what you get in evaluating the expression for 2) is equal to 0.

Now look at the third row and use the lead coefficient of 2 and final entry of 15 (ignore the 0). You can now limit your choices to only factors of +15 divided by factors of 2. The new, revised list is: $\pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}$, and $\pm \frac{15}{2}$.

I would probably try only integers before trying any fractions, but I want you to see what using a fraction in synthetic division looks like. I choose to try $-\frac{1}{2}$. Use only the numbers in the last row of the previous division.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 17 & 38 & 15 \\ & & -1 & -8 & -15 \\ \hline & 2 & 16 & 30 & 0 \end{array}$$

Such a wise choice! The number worked and is a solution. You could go on with more synthetic division, but, at this point, I usually stop. The first three numbers in the bottom row represent a quadratic trinomial. Write out the trinomial, factor it, use the MPZ, and find the last two solutions.

The quadratic equation represented by that last row is: $2x^2 + 16x + 30 = 0$.

First factor 2 out of each term. Then factor the trinomial: $2(x^2 + 8x + 15) = 2(x + 3)(x + 5) = 0$.

The solutions from the factored trinomial are $x = -3$ and $x = -5$. Add these two solutions to $x = 2$ and $x = -\frac{1}{2}$, and you have the four solutions of the polynomial.

What? You're miffed! You wanted me to finish the problem using synthetic division — not bail out and factor? Okay. I'll pick up where I left off with the synthetic division and show you how it finishes:

$$\begin{array}{r|rrr} -3 & 2 & 16 & 30 \\ & & -6 & -30 \\ \hline & 2 & 10 & 0 \end{array}$$

And, finally:

$$\begin{array}{r|rr} -5 & 2 & 10 \\ & & -10 \\ \hline & 2 & 0 \end{array}$$

Working Quadratic-Like Equations

Some equations with higher powers or fractional powers are *quadratic-like,* meaning that they have three terms and

- The variable in the first term has an even power (4, 6, 8, . . .) or $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$.
- The variable in the second term has a power that is half that of the first.
- The third term is a constant number.

In general, the format for a quadratic-like equation is: $ax^{2n} + bx^n + c = 0$. Just as in the general quadratic equation, the x is the variable and the a, b, and c are constant numbers. The a can't be 0, but the other two letters have no restrictions. (I show you how to factor quadratic-like expressions in Chapter 5.)

To solve a quadratic-like equation, first pretend that it's quadratic, and use the same methods as you do for those; and then do a step or two more. The extra steps usually involve taking an extra root or raising to an extra power.

Now I'll show you the steps used to solve quadratic-like equations by working through a couple of examples.

Solve for x in $x^4 - 5x^2 + 4 = 0$.

1. **Rewrite the equation, replacing the actual powers with the numbers 2 and 1.**

   Rewrite this as a quadratic equation using the same *coefficients* (number multipliers) and constant.

   Change the letter used for the variable, so you won't confuse this new equation with the original. Substitute q for x^2 and q^2 for x^4:

   $$q^2 - 5q + 4 = 0$$

2. **Factor the quadratic equation.**

   $q^2 - 5q + 4 = 0$ factors nicely into $(q - 4)(q - 1) = 0$.

3. **Reverse the substitution and use the factorization pattern to factor the original equation.**

   Use that same pattern to write the factorization of the original problem. When you replace the variable q in the factored form, use x^2:

   $$x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1) = 0$$

4. **Solve the equation using the MPZ.**

   Either $x^2 - 4 = 0$ or $x^2 - 1 = 0$. If $x^2 - 4 = 0$, then $x^2 = 4$ and $x = \pm 2$. If $x^2 - 1 = 0$, then $x^2 = 1$ and $x = \pm 1$.

This fourth-degree equation did live up to its reputation and have four different solutions.

This next example presents an interesting problem because the exponents are fractions. But the trinomial fits into the category of *quadratic-like,* so I'll show you how you can take advantage of this format to solve the equation. And, no, the rule of the number of solutions doesn't work the same way here. There aren't any possible situations where there's half a solution.

EXAMPLE

Solve $w^{\frac{1}{2}} - 7w^{\frac{1}{4}} + 12 = 0$.

1. **Rewrite the equation with powers of 2 and 1. Substitute q for $w^{\frac{1}{4}}$ and q^2 for $w^{\frac{1}{2}}$.**

   Remember: Squaring $w^{\frac{1}{4}}$ gives you $\left(w^{\frac{1}{4}}\right)^2 = w^{\frac{2}{4}} = w^{\frac{1}{2}}$.

   Rewrite the equation as $q^2 - 7q + 12 = 0$.

2. **Factor.**

   This factors nicely into $(q - 3)(q - 4) = 0$.

3. **Replace the variables from the original equation, using the pattern.**

   Replace with the original variables to get $\left(w^{\frac{1}{4}} - 3\right)\left(w^{\frac{1}{4}} - 4\right) = 0$.

4. **Solve the equation for the original variable, w.**

   $$\left(w^{\frac{1}{4}} - 3\right)\left(w^{\frac{1}{4}} - 4\right) = 0$$

Now, when you use the MPZ, you get that either $w^{\frac{1}{4}} - 3 = 0$ or $w^{\frac{1}{4}} - 4 = 0$. How do you solve these things? Look at $w^{\frac{1}{4}} - 3 = 0$. Adding 3 to each side, you get $w^{\frac{1}{4}} = 3$. You can solve for w if you raise each side to the fourth power: $\left(w^{\frac{1}{4}}\right)^4 = (3)^4$. This says that $w = 81$. Doing the same with the other factor, if $w^{\frac{1}{4}} - 4 = 0$, then $w^{\frac{1}{4}} = 4$ and $\left(w^{\frac{1}{4}}\right)^4 = (4)^4$. This says that $w = 256$.

5. **Check the answers.**

   If $w = 81$, $(81)^{\frac{1}{2}} - 7(81)^{\frac{1}{4}} + 12 = 9 - 7(3) + 12 = 21 - 21 = 0$.

   If $w = 256$,
   $(256)^{\frac{1}{2}} - 7(256)^{\frac{1}{4}} + 12 = 16 - 7(4) + 12 = 28 - 28 = 0$.

   They both work.

Negative exponents are another interesting twist to these equations, as you see in the next example.

EXAMPLE

Solve for the value of x in $2x^{-6} - x^{-3} - 3 = 0$.

1. **Rewrite the equation using powers of 2 and 1. Substitute q for x^{-3} and q^2 for x^{-6}.**

   Rewrite the equation as $2q^2 - q - 3 = 0$.

2. **Factor.**

   This factors into $(2q - 3)(q + 1) = 0$.

3. **Go back to the original variables and powers.**

   Use this pattern. Factor the original equation to get:

   $$(2x^{-3} - 3)(x^{-3} + 1) = 0$$

4. **Solve.**

   Use the MPZ. The two equations to solve are $2x^{-3} - 3 = 0$ and $x^{-3} + 1 = 0$. These become $2x^{-3} = 3$ and $x^{-3} = -1$. Rewrite these using the definition of negative exponents:

   $$x^{-n} = \frac{1}{x^n}$$

   So the two equations can be written $\frac{2}{x^3} = 3$ and $\frac{1}{x^3} = -1$. Cross-multiply in each case to get $3x^3 = 2$

and $x^3 = -1$. Divide the first equation through by 3 to get the x^3 alone, and then take the cube root of each side to solve for x:

$$x = \sqrt[3]{\frac{2}{3}} \text{ or } x = \sqrt[3]{-1} = -1$$

Rooting Out Radicals

Some equations have radicals in them. You change those equations to linear or quadratic equations for greater convenience when solving. A basic process that leads to a solution of equations involving a radical involves getting rid of that radical. Removing the radical changes the problem into something more manageable, but the change also introduces the possibility of a nonsense answer or an error. Checking your answer is even more important in the case of solving radical equations.

The main method to use when dealing with equations that contain radicals is to change the equations to those that do not have radicals in them. You accomplish this by raising the radical to a power that changes the fractional exponent (representing the radical) to a 1.

Raising to powers clears out the radicals, but problems can occur when the variables are raised to even powers. Variables can stand for negative numbers or values that allow negatives under the radical, which isn't always apparent until you get into the problem and check an answer. Instead of going on with all this doom and gloom and the problems that occur when powering up both sides of an equation, let me show you some examples of how the process works, what the pitfalls are, and how to deal with any extraneous solutions.

Solve for x in $2\sqrt{x+15} - 3 = 9$.

1. **Get the radical term by itself on one side of the equation.**

   The first step is to add 3 to each side: $2\sqrt{x+15} = 12$.

2. **Square both sides of the equation. (You could divide both sides by 2, but I want to show you an important rule when squaring both sides.)**

One of the rules involving exponents is the square of the product of two factors is equal to the product of each of those same factors squared: $(a \cdot b)^2 = a^2 \cdot b^2$.

Squaring the left side,
$\left(2\sqrt{x+15}\right)^2 = 2^2 \cdot \left(\sqrt{x+15}\right)^2 = 4\left(x+15\right)$.

Squaring the right side, $12^2 = 144$.

So you get the new equation: $4\ (x + 15) = 144$.

3. **Solve for *x* in the new, linear equation.**

   Distribute the 4, first: $4x + 60 = 144$.

   Subtract 60 from each side, and you get $4x = 84$ or $x = 21$.

4. **Check your work.**

$$\begin{aligned} 2\sqrt{x+15} - 3 &= 9 \\ 2\sqrt{21+15} - 3 &= 2\sqrt{36} - 3 \\ &= 2 \cdot 6 - 3 \\ &= 12 - 3 \\ &= 9 \end{aligned}$$

Next I show you an example where you find two different solutions, but only one of them works.

Solve for z in $7 + \sqrt{z-1} = z$.

1. **Get the radical by itself on the left.**

   Subtracting 7 from each side, you end up with the radical on the left and a binomial on the right.

$$\sqrt{z-1} = z - 7$$

2. **Square both sides of the equation.**

   The only thing to watch out for here is squaring the binomial correctly.

$$\begin{aligned} \left(\sqrt{z-1}\right)^2 &= \left(z-7\right)^2 \\ z - 1 &= z^2 - 14z + 49 \end{aligned}$$

3. **Solve the equation.**

   This time you have a quadratic equation. Move everything over to the right, so that you can set the equation equal to 0. To do this, subtract z from each side and add 1 to each side.

   $$0 = z^2 - 15z + 50$$

   The right side factors, giving you $(z - 5)(z - 10) = 0$. Using the MPZ, you get either $z = 5$ or $z = 10$.

4. **Check your answer.**

   Check these carefully because incorrect answers often show up — especially when you create a quadratic equation by the squaring-each-side process.

   If $z = 5$, then $7 + \sqrt{5-1} = 7 + \sqrt{4} = 7 + 2 = 9 \neq 5$. The 5 doesn't work.

   If $z = 10$, then $7 + \sqrt{10-1} = 7 + \sqrt{9} = 7 + 3 = 10$. The 10 does work.

The only solution is that z equals 10. That's fine. Sometimes these problems have two answers, sometimes just one answer, or sometimes no answer at all. The method works — you just have to be careful.